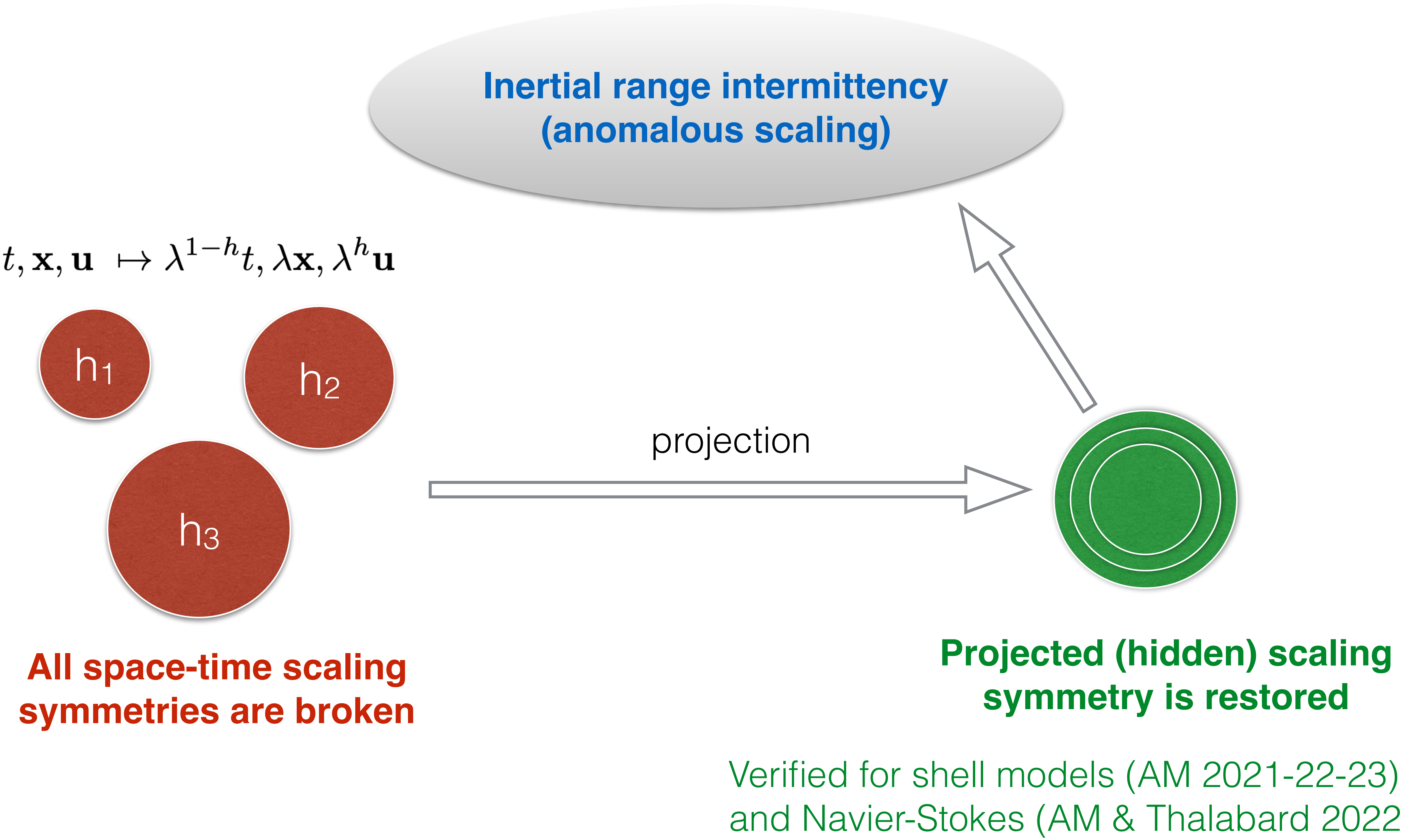
Hidden scale invariance of turbulence at dissipation scales

Alexei A. Mailybaev

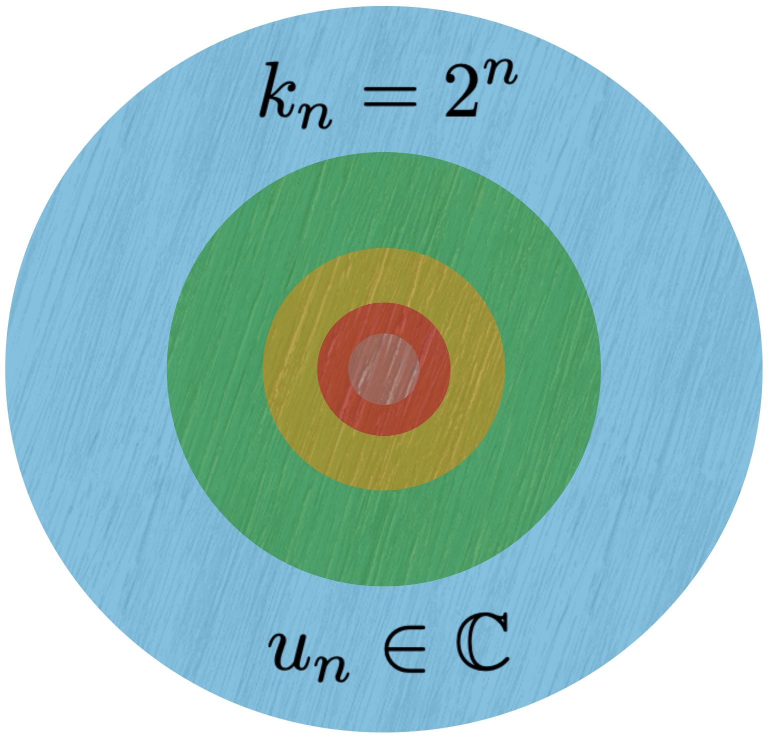
IMPA, Rio de Janeiro



Question addressed this work:

What is a relation between the hidden symmetry and dissipation?

Shell model (Sabra)



Equations: $$\frac{du_n}{dt} = k_0 \mathcal{B}_n[u] - \nu k_n^2 u_n$$

Quadratic “convective" term:

$$\mathcal{B}_n[u] = i2^n \left(2u_{n+2}u_{n+1}^* - \frac{u_{n+1}u_{n-1}^*}{2} + \frac{u_{n-1}u_{n-2}}{4} \right)$$

Boundary/forcing conditions: $u_0(t) \equiv u_0 > 0, \quad u_n(t) \equiv 0 \;\text{ for }\; n < 0$

Scale invariance: $t,\ u_n,\ \nu \;\mapsto\; 2^{1-h}t,\ 2^h u_{n+1},\ 2^{1+h}\nu.$

Inviscid conserved quantities:

$\mathcal{E}[u] = \frac{1}{2}\sum_n |u_n|^2$ (energy)

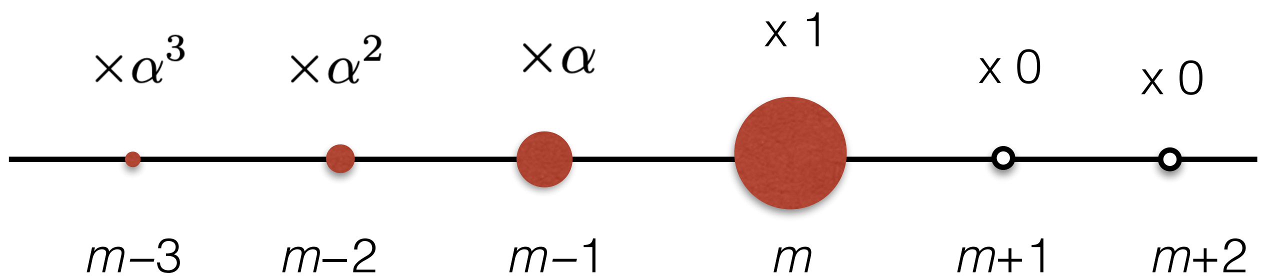
$\mathcal{H}[u] = \sum_n (-1)^n k_n |u_n|^2$ (helicity)

Reynolds number: $\mathrm{R} = u_0 \ell_0 / \nu$

Rescaled (projected) system

Local velocity amplitude and turnover time:

$$\mathcal{A}_m[u] = \sqrt{\sum_{j\geq 0} \alpha^j |u_{m-j}|^2}, \quad \mathcal{T}_m[u] = \frac{\ell_m}{\mathcal{A}_m[u]}. \qquad (0 < \alpha < 0.4)$$



with a fixed reference scale $m \geq 0$

Rescaled velocities and time: $U_N^{(m)} = \dfrac{u_{m+N}}{\mathcal{A}_m[u]}, \quad d\tau^{(m)} = \dfrac{dt}{\mathcal{T}_m[u]}$ (projected variables)

Rescaled equations of motion:

$$\frac{dU_N^{(m)}}{d\tau^{(m)}} = \mathcal{B}_N[U^{(m)}] - U_N^{(m)} \sum_{j=0}^{m-1} \alpha^j \mathrm{Re}\left(U_{-j}^{(m)*}\mathcal{B}_{-j}[U^{(m)}]\right)$$

$$- \frac{U_N}{\mathrm{R}_m[u]}\left(4^N - \sum_{j=0}^{m-1} \alpha^j 4^{-j} |U_{-j}|^2\right), \quad N > -m.$$

$$\mathrm{R}_m[u] = \frac{\mathcal{A}_m[u]\ell_m}{\nu}.$$

(Local Reynolds)

+ B.C.

Closed form of the nonlinear terms

Dissipation term is not closed in terms of new variables

Inertial interval

Hidden symmetry in the ideal rescaled system

$$\frac{dU_N}{d\tau} = \mathcal{B}_N[U] - U_N \sum_{j \geq 0} \alpha^j \mathrm{Re}\left(U_{-j}^* \mathcal{B}_{-j}[U]\right), \quad N \in \mathbb{Z},$$

(no viscosity, no B.C.)

Hidden scaling symmetry:

$$\hat{U}_N = \frac{U_{N+1}}{\sqrt{\alpha + |U_1|^2}}, \quad d\hat{\tau} = 2\sqrt{\alpha + |U_1|^2}\, d\tau. \quad (m \mapsto m+1)$$

Hidden scale invariance is restored statistically in the inertial range.

Kolmogorov multipliers:

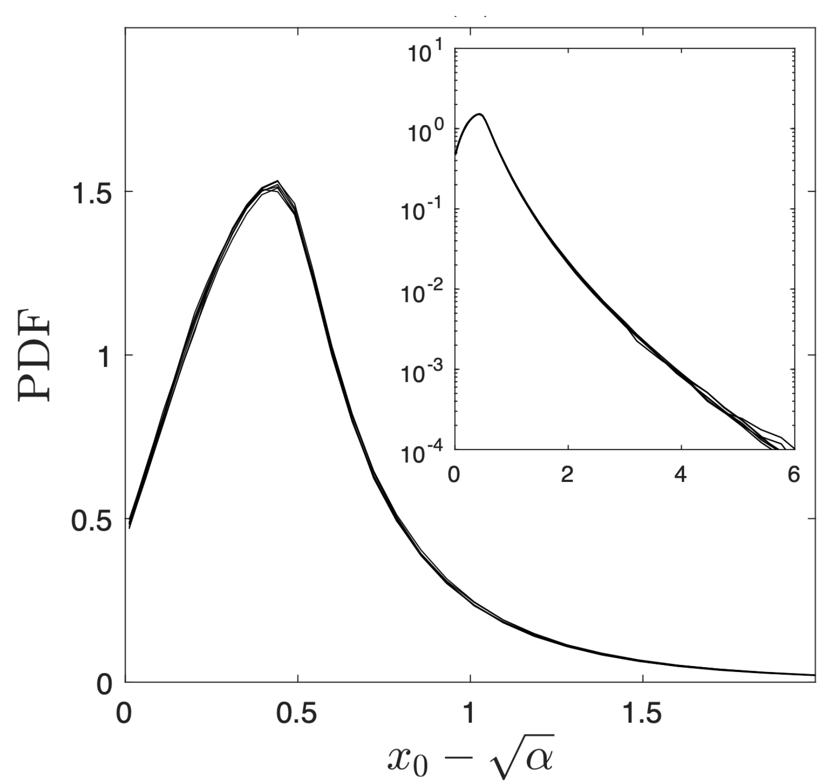
$$\mathcal{X}_N[U^{(m)}] = \frac{\mathcal{A}_n[u]}{\mathcal{A}_{n-1}[u]} = \sqrt{\alpha + \frac{\alpha |U_N^{(m)}|^2}{\sum_{j \geq 1} \alpha^j |U_{N-j}^{(m)}|^2}}, \quad n = m + N$$

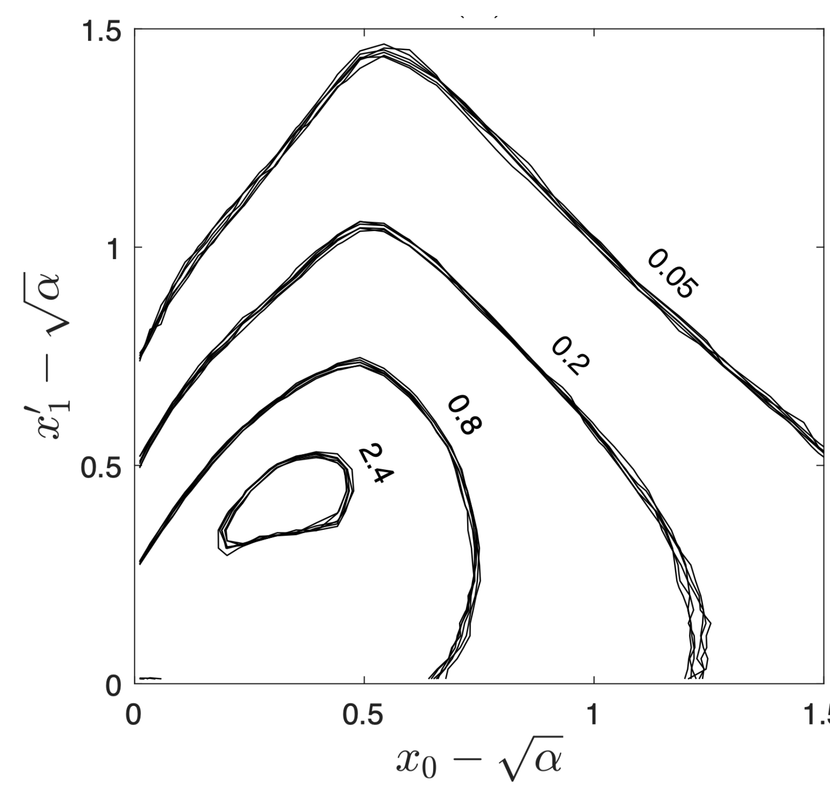
$$x_0 = \mathcal{X}_0[U^{(m)}(\tau^{(m)})]$$

$$x_1' = \mathcal{X}_1[U^{(m)}(\tau^{(m)}+1)]$$

$$m = 7, \ldots, 12$$

$$\mathrm{R} = 10^{10}$$

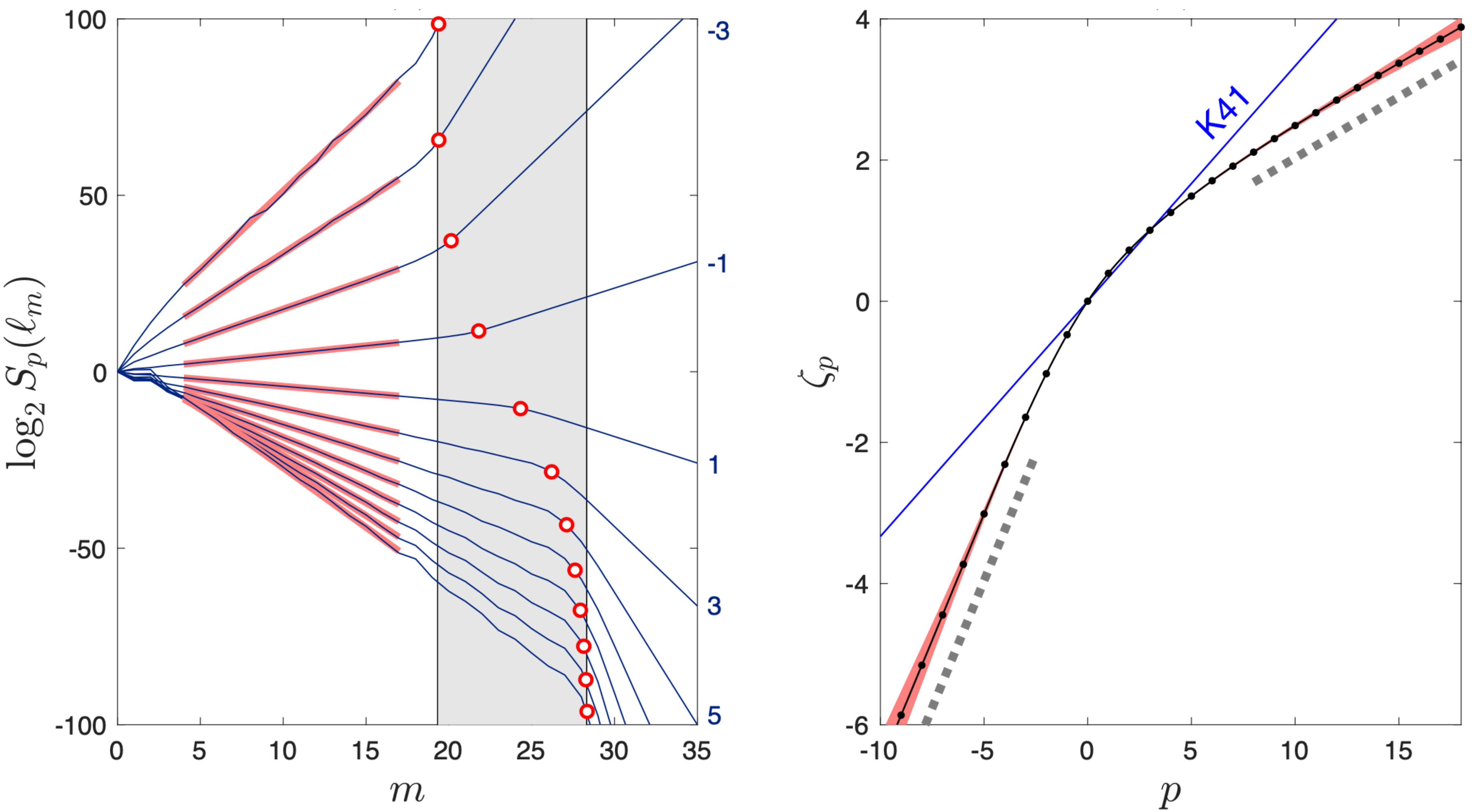




Structure functions: $S_p(\ell_m) = \langle \mathcal{A}_m^p[u] \rangle_t \,, \quad p \in \mathbb{R}, \quad \ell_m = 1/k_m$

Power-law scaling: $S_p(\ell_m) \propto \ell_m^{\zeta_p}$

Intermittency/anomaly: nonlinear dependence of scaling exponents on the order *p*



Power-law scalings as Perron-Frobenius modes

Structure functions: $S_p(\ell_m) = \langle \mathcal{A}_m^p[u] \rangle_t \,, \quad \mathcal{A}_m[u] = u_0 \prod_{j=0}^{m-1} x_{-j}, \quad x_j = \mathcal{X}_j[U^{(m)}]$

Recurrent expressions in terms of rescaled variables:

$$S_p(\ell_m) = u_0^p \int d\mu_p^{(m)}(x_0, x_{-1}, \ldots)$$

$$d\mu_p^{(m)} = \mathcal{L}_p^{(m-1)} \circ \mathcal{L}_p^{(m-2)} \circ \cdots \circ \mathcal{L}_p^{(1)}[d\mu_p^{(1)}]$$

Positive linear operator: $\mathcal{L}_p^{(m)}[d\mu] = x_0^p \, p^{(m)}(x_0|x_{-1}, x_{-2}, \ldots) \, dx_0 \, d\mu(x_{-1}, x_{-2}, \ldots)$

Hidden symmetry in the inertial range: $\mathcal{L}_p^{(m)} \approx \Lambda_p$ (independent of *m*)

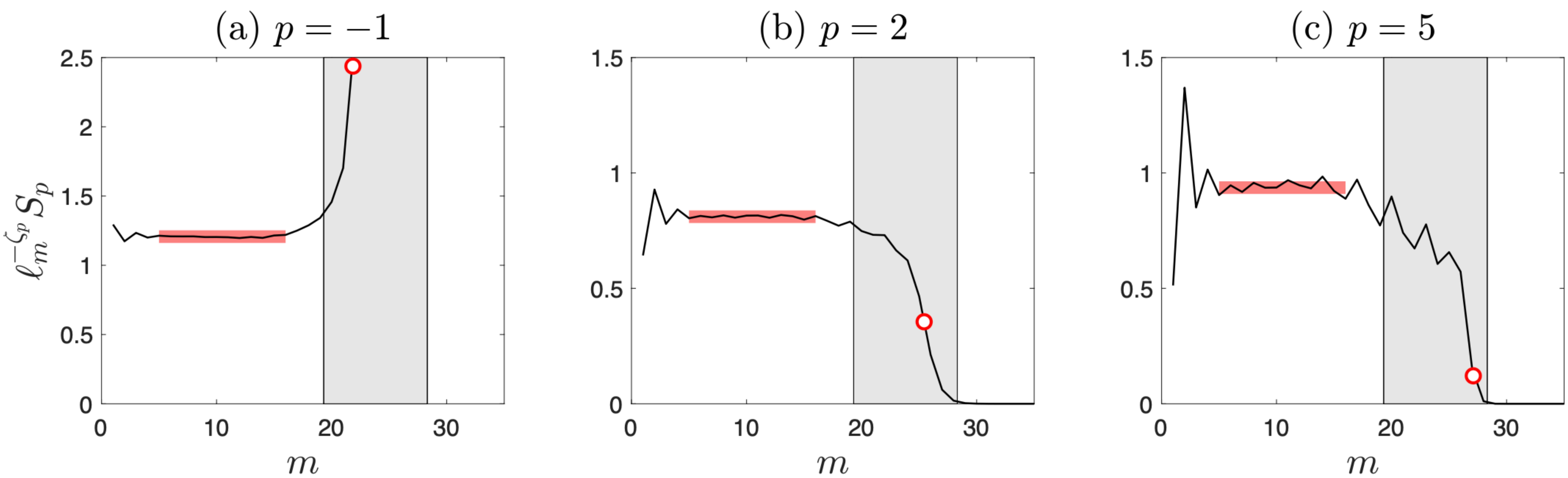
Perron-Frobenius eigenmode: $d\mu_p^{(m)} \approx C_p \lambda_p^m \, d\nu_p, \quad \Lambda_p[d\nu_p] = \lambda_p \, d\nu_p.$

Scaling exponents: $S_p(\ell_m) \approx C_p u_0^p \left(\frac{\ell_m}{\ell_0} \right)^{\zeta_p}$

$$\boxed{\zeta_p = -\log_2 \lambda_p}$$

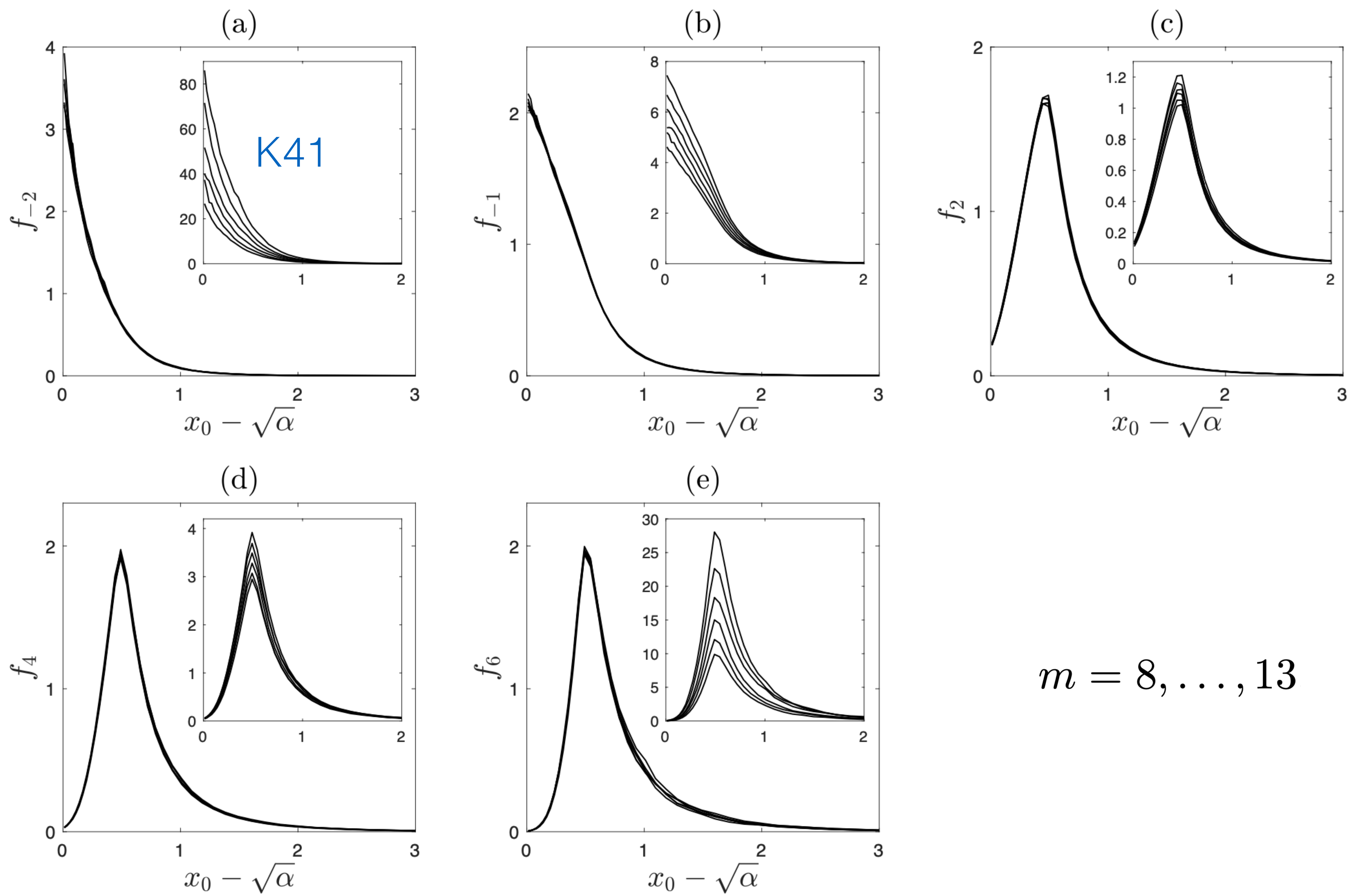
Compensated structure functions

$$S_p(\ell_m) \approx C_p u_0^p \left(\frac{\ell_m}{\ell_0}\right)^{\zeta_p} \quad \longrightarrow \quad \ell_m^{-\zeta_p} S_p \quad \text{is constant}$$



Perron-Frobenius eigenmodes

$$d\mu_p^{(m)} \approx C_p \lambda_p^m \, d\nu_p, \quad \longrightarrow \quad d\nu_p \approx \frac{1}{C_p}\left(\frac{\ell_m}{\ell_0}\right)^{-\zeta_p} d\mu_p^{(m)}.$$



$$m = 8, \ldots, 13$$

Large deviation principle and multifractality

Logarithm of a multiplier: $$w_n = -\log_2 \mathcal{X}_{n-m}[U^{(m)}] = -\log_2 \frac{\mathcal{A}_n[u]}{\mathcal{A}_{n-1}[u]}$$

Sample mean: $$W_m = \frac{w_1 + w_2 + \cdots + w_m}{m}$$

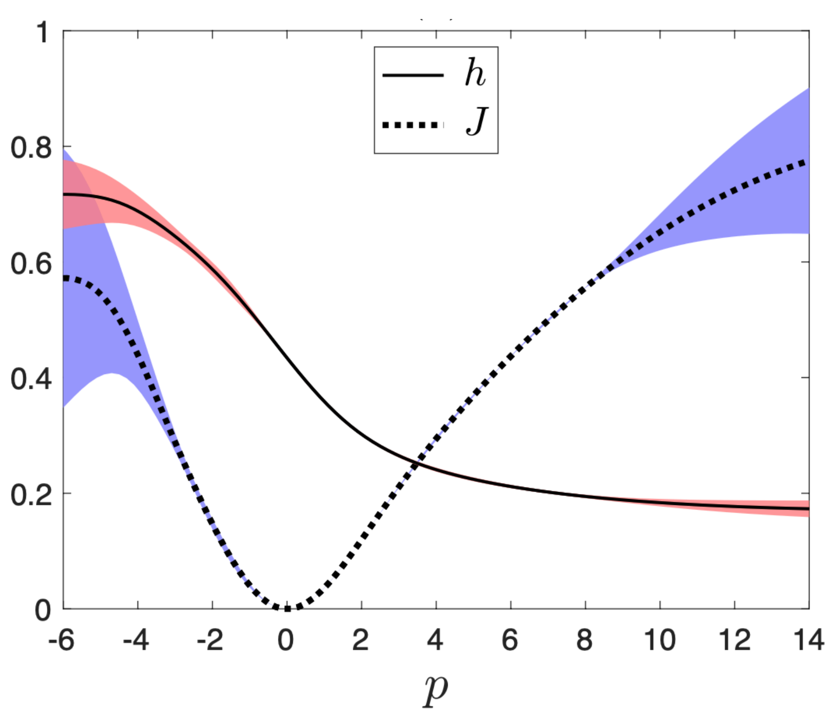
Moments: $$\frac{\langle \mathcal{A}_m^p[u] \rangle_t}{u_0^p} = \left\langle 2^{-mpW_m} \right\rangle_t \approx 2^{-m\zeta_p} C_p$$

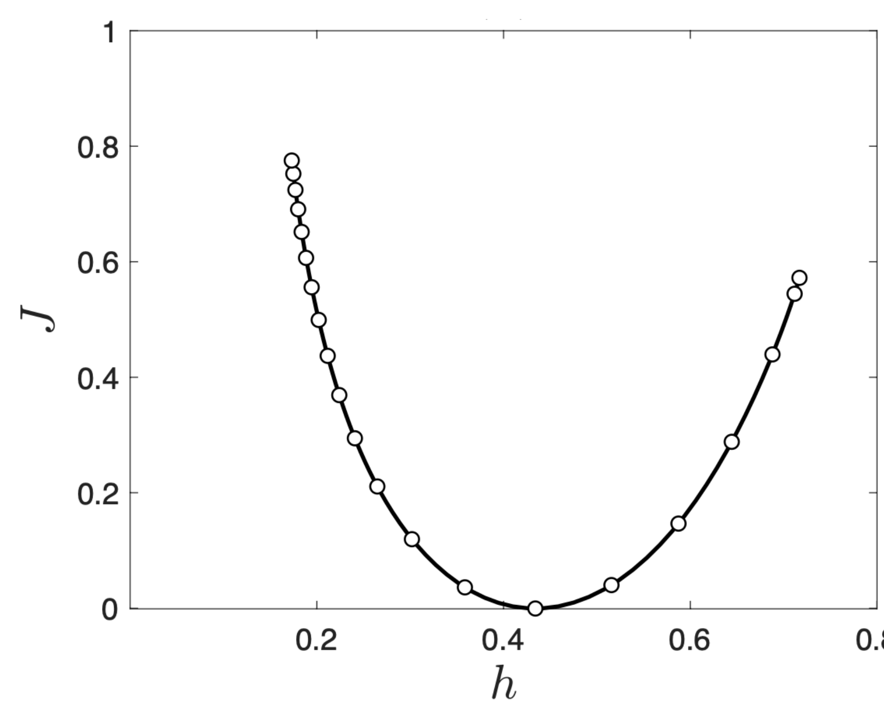
Large Deviation Principle (Gartner-Ellis Theorem): $$P\left(W_m \in [h, h+dh]\right) \approx 2^{-mJ(h)}\, dh.$$

Rate (Cramer) function: $$J(h) = \sup_{p \in \mathbb{R}} \left(\zeta_p - ph\right)$$

Fractal codimension of h-Holder-continuous subset

In original variables: $$\mathcal{A}_m[u] \sim u_0 \left(\frac{\ell_m}{\ell_0}\right)^h \quad \text{with probability} \quad P \sim \left(\frac{\ell_m}{\ell_0}\right)^{J(h)}$$





$$h_{\min} \le h \le h_{\max},$$

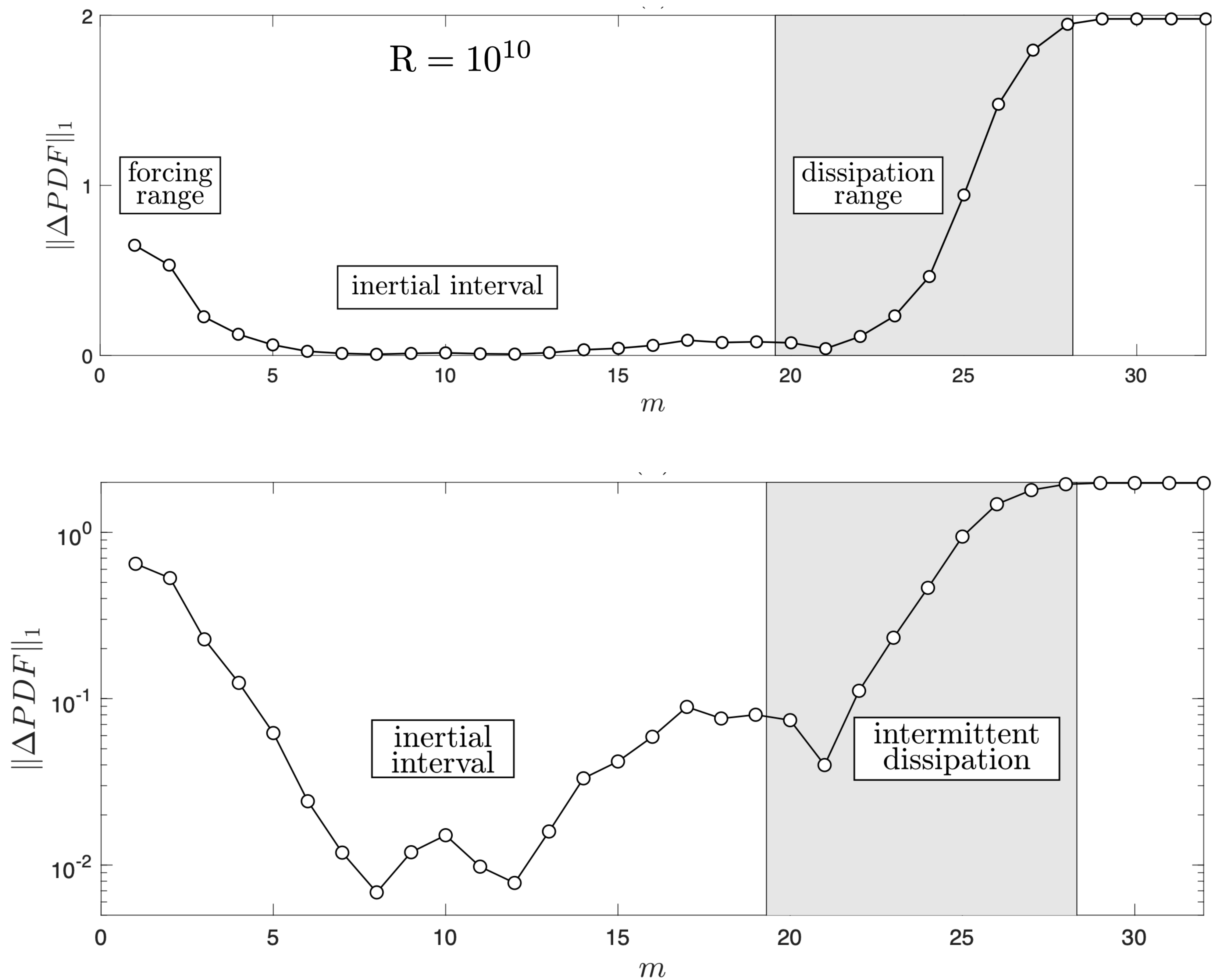
$$h_{\min} \approx 0.173 \pm 0.015,$$

$$h_{\max} \approx 0.72 \pm 0.06.$$

Forcing and dissipation ranges

Global view on the hidden symmetry

Deviation from the hidden-symmetric PDF for a multiplier $x_0 = \mathcal{X}_0[U^{(m)}]$



Dissipation range

Rescaled equations of motion:

$$\frac{dU_N^{(m)}}{d\tau^{(m)}} = \mathcal{B}_N[U^{(m)}] - U_N^{(m)} \sum_{j=0}^{m-1} \alpha^j \mathrm{Re}\left(U_{-j}^{(m)*} \mathcal{B}_{-j}[U^{(m)}]\right)$$

$$-\frac{U_N}{\mathrm{R}_m[u]}\left(4^N - \sum_{j=0}^{m-1} \alpha^j 4^{-j} |U_{-j}|^2\right), \quad N > -m.$$

$$\mathrm{R}_m[u] = \frac{\mathcal{A}_m[u]\ell_m}{\nu}.$$

(Local Reynolds)

Closed form for nonlinear terms

Dissipation term is not closed in terms of new variables

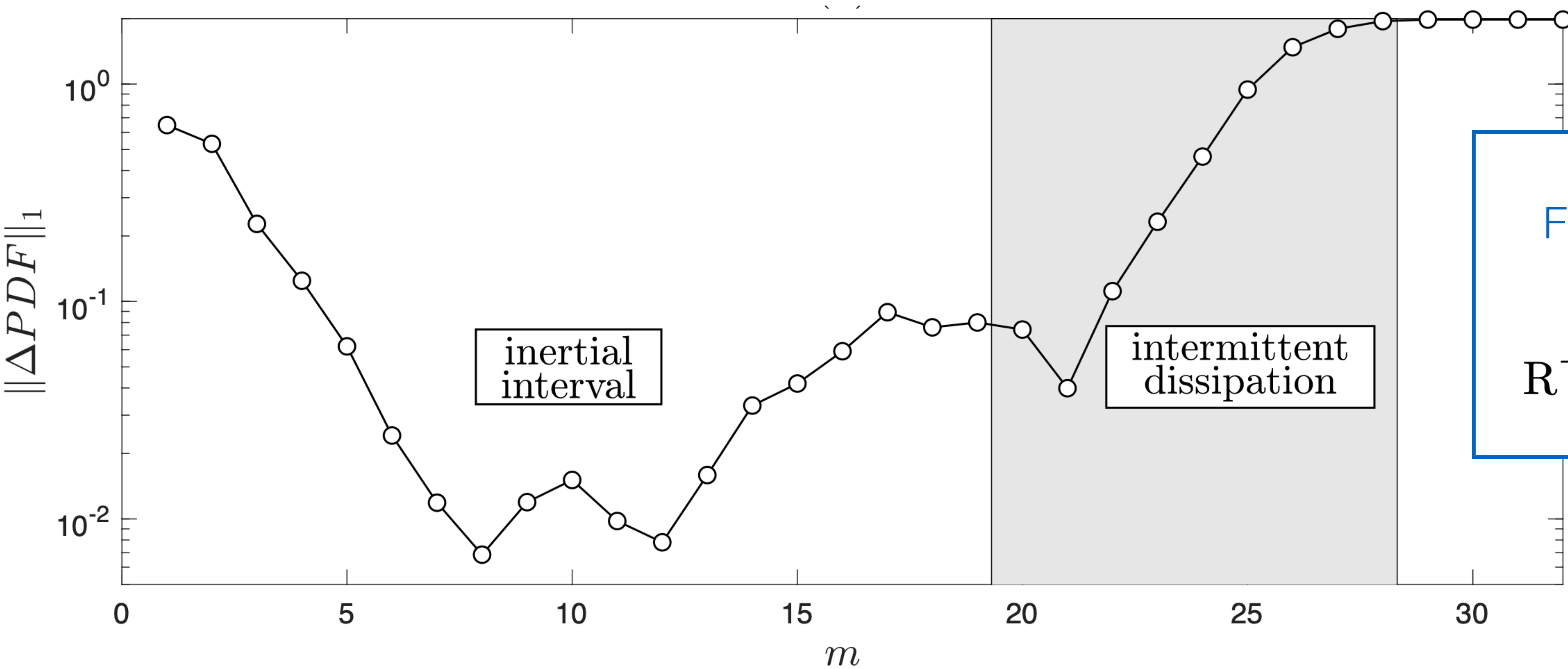
Intermittent dissipation!

Large Deviation Principle (analysis following Frisch & Vergassola 1991) :

$$\mathrm{R}_m[u] \sim \frac{\ell_m u_0}{\nu}\left(\frac{\ell_m}{\ell_0}\right)^h = \mathrm{R}\left(\frac{\ell_m}{\ell_0}\right)^{1+h} \quad \text{with probability} \quad P \sim \left(\frac{\ell_m}{\ell_0}\right)^{J(h)}$$

Negligible dissipation: $\frac{\ell_m}{\ell_0} \gg \mathrm{R}^{-1/(1+h_{\max})}$ $\qquad h_{\max} \approx 0.72 \pm 0.06.$

Dominant dissipation: $\frac{\ell_m}{\ell_0} \ll \mathrm{R}^{-\frac{1}{1+h_{\min}}}$ $\qquad h_{\min} \approx 0.173 \pm 0.015,$



Far dissipation range

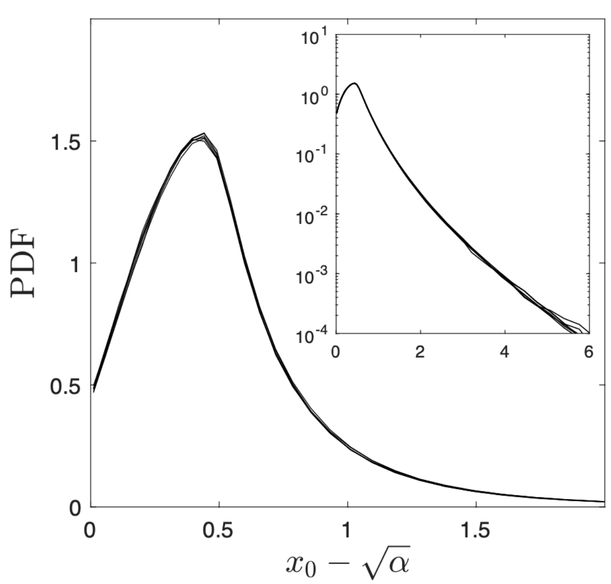
$$\mathrm{R}^{-\frac{1}{1+h_{\min}}} \gg \frac{\ell_m}{\ell_0}$$

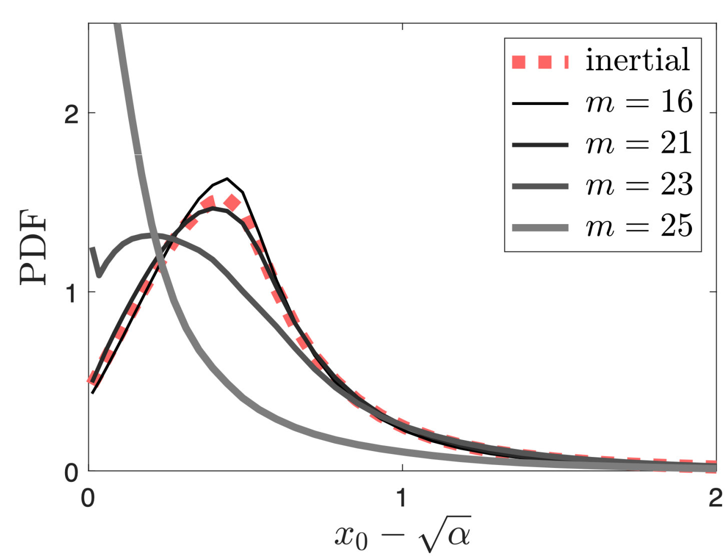
Inertial interval

$$1 \gg \frac{\ell_m}{\ell_0} \gg \mathrm{R}^{-\frac{1}{1+h_{\max}}}$$

Intermittent dissipation range

$$\mathrm{R}^{-\frac{1}{1+h_{\max}}} \gtrsim \frac{\ell_m}{\ell_0} \gtrsim \mathrm{R}^{-\frac{1}{1+h_{\min}}}$$





Structure functions in the intermittent dissipation range

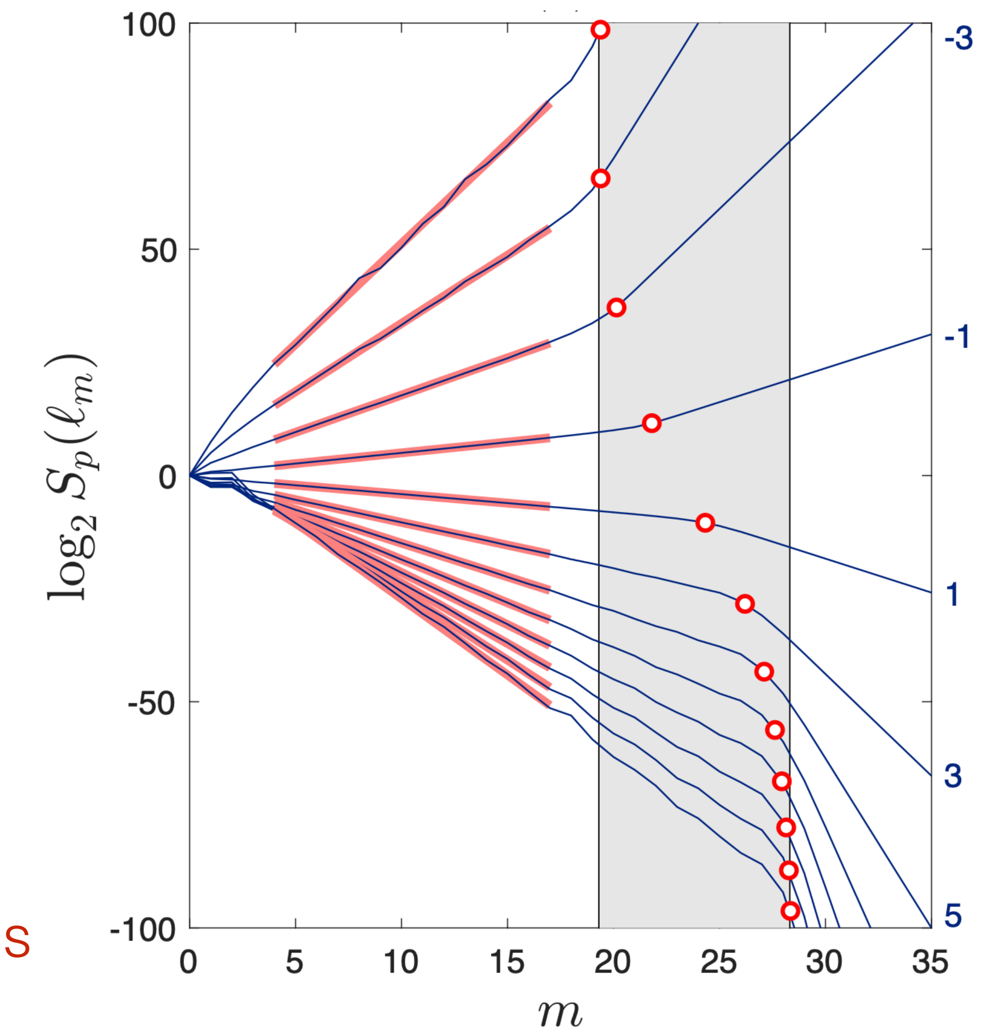
$$\langle \mathcal{A}_m^p[u] \rangle_t \sim u_0^p \int \left(\frac{\ell_m}{\ell_0} \right)^{ph+J(h)} d\mu(h)$$

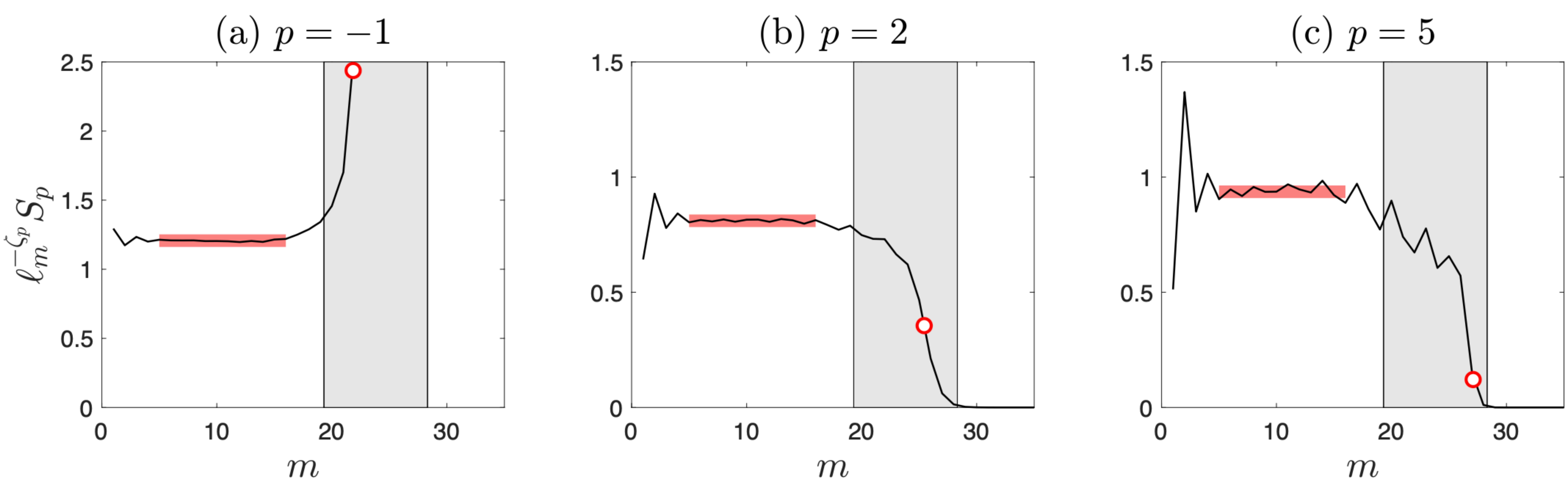
Frisch & Vergassola
1991

It is dominated by $\mathcal{A}_m[u] \sim u_0 \left(\frac{\ell_m}{\ell_0} \right)^{H(p)}, \quad H(p) = \frac{d\zeta_p}{dp}$

Order-dependent viscous scale: $\ell_m \gg \eta(p) = \mathrm{R}^{-\frac{1}{1+H(p)}} \ell_0$

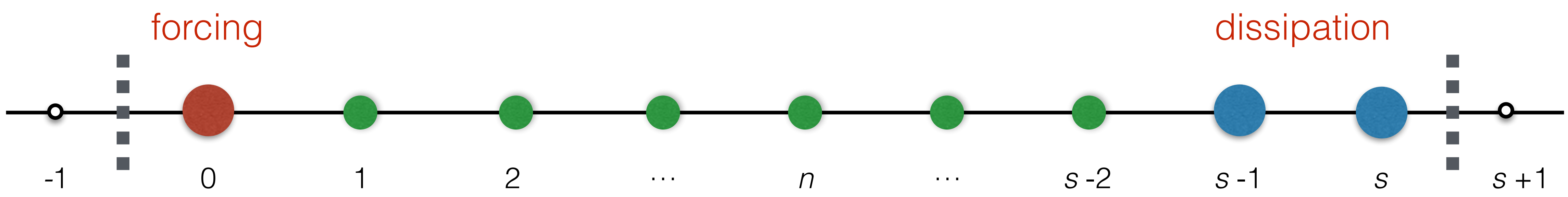
But subsets with different *H*(*p*) interact!
Hence, we expect that the dissipation affects structure functions at all scales of the intermittent dissipation range.





Hidden-symmetric dissipation

Shell model with a viscous cutoff



Equations of motion: $$\frac{du_n}{dt} = k_0 \mathcal{B}_n^{(s)}[u], \quad n = 1, \ldots, s$$

Quadratic terms: $$\mathcal{B}_n^{(s)}[u] = \mathcal{B}_n[u] + \begin{cases} -2^n |u_n| u_n, & n = s-1, s; \\ 0, & \text{otherwise.} \end{cases}$$ dissipation

$$\mathcal{B}_n[u] = i2^n \left(2u_{n+2}u_{n+1}^* - \frac{u_{n+1}u_{n-1}^*}{2} + \frac{u_{n-1}u_{n-2}}{4} \right)$$

Boundary and cutoff conditions: $$u_0(t) \equiv u_0 > 0, \quad u_n(t) \equiv 0 \ \text{ for } \ n \notin \{0, 1, \ldots, s\}$$

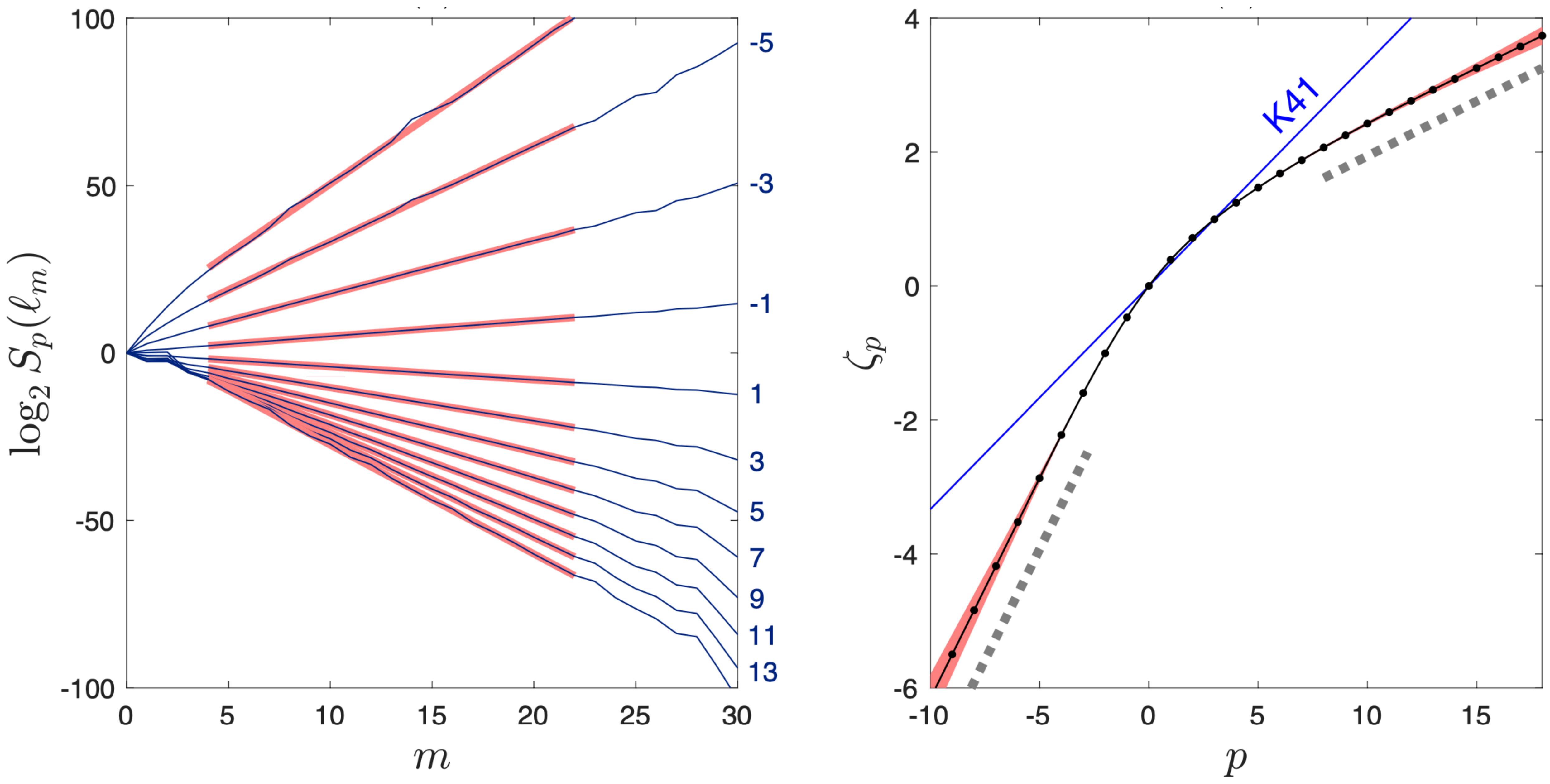
Scale invariance at small scales: $$t,\ u_n,\ s \ \mapsto \ 2^{1-h}t,\ 2^h u_{n+1},\ s-1$$

Instead of $t,\ u_n,\ \nu \ \mapsto \ 2^{1-h}t,\ 2^h u_{n+1},\ 2^{1+h}\nu.$

Structure functions: $S_p(\ell_m) = \langle \mathcal{A}_m^p[u] \rangle_t \,, \quad p \in \mathbb{R}, \quad \ell_m = 1/k_m$

Power-law scaling: $S_p(\ell_m) \propto \ell_m^{\zeta_p}$

Intermittency/anomaly: nonlinear dependence of scaling exponents on the order *p*



Extended hidden symmetry at small scales

Local velocity amplitude and turnover time:
$$\mathcal{A}_m[u] = \sqrt{\sum_{j\geq 0} \alpha^j |u_{m-j}|^2}, \quad \mathcal{T}_m[u] = \frac{\ell_m}{\mathcal{A}_m[u]}. \qquad (0 < \alpha < 0.4)$$

Rescaled velocities and time:
$$U_N^{(m)} = \frac{u_{m+N}}{\mathcal{A}_m[u]}, \quad d\tau^{(m)} = \frac{dt}{\mathcal{T}_m[u]}$$
(projected variables)

Rescaled equations of motion (neglecting large-scale boundary effects):
$$\frac{dU_N}{d\tau} = \mathcal{B}_N^{(S)}[U] - U_N \sum_{j\geq 0} \alpha^j \mathrm{Re}\left(U_{-j}^* \mathcal{B}_{-j}^{(S)}[U]\right), \qquad -m < N \leq S, \qquad S = s - m.$$

Cutoff: $U_N^{(m)} \equiv 0, \quad N > S.$

Extended hidden symmetry:
$$\boxed{U_N,\ d\tau,\ S\ \mapsto\ \frac{U_{N+1}}{\sqrt{\alpha + |U_1|^2}},\ 2\sqrt{\alpha + |U_1|^2}\, d\tau,\ S.}$$

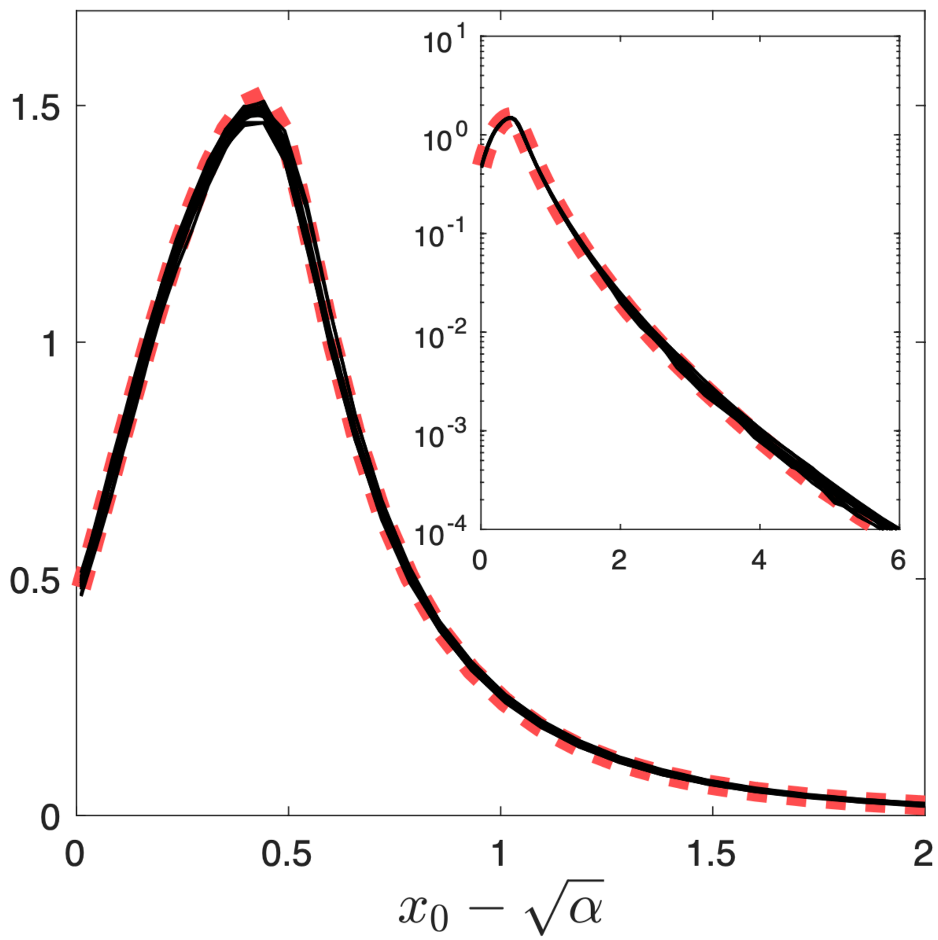
change of the reference shell $m \mapsto m + 1$

simultaneously with the cutoff shell $s \mapsto s + 1$

Statistics of Kolmogorov multipliers

Kolmogorov multipliers: $$\mathcal{X}_N[U^{(m)}] = \frac{\mathcal{A}_n[u]}{\mathcal{A}_{n-1}[u]} = \sqrt{\alpha + \frac{\alpha |U_N^{(m)}|^2}{\sum_{j\geq 1} \alpha^j |U_{N-j}^{(m)}|^2}}, \quad n = m + N$$

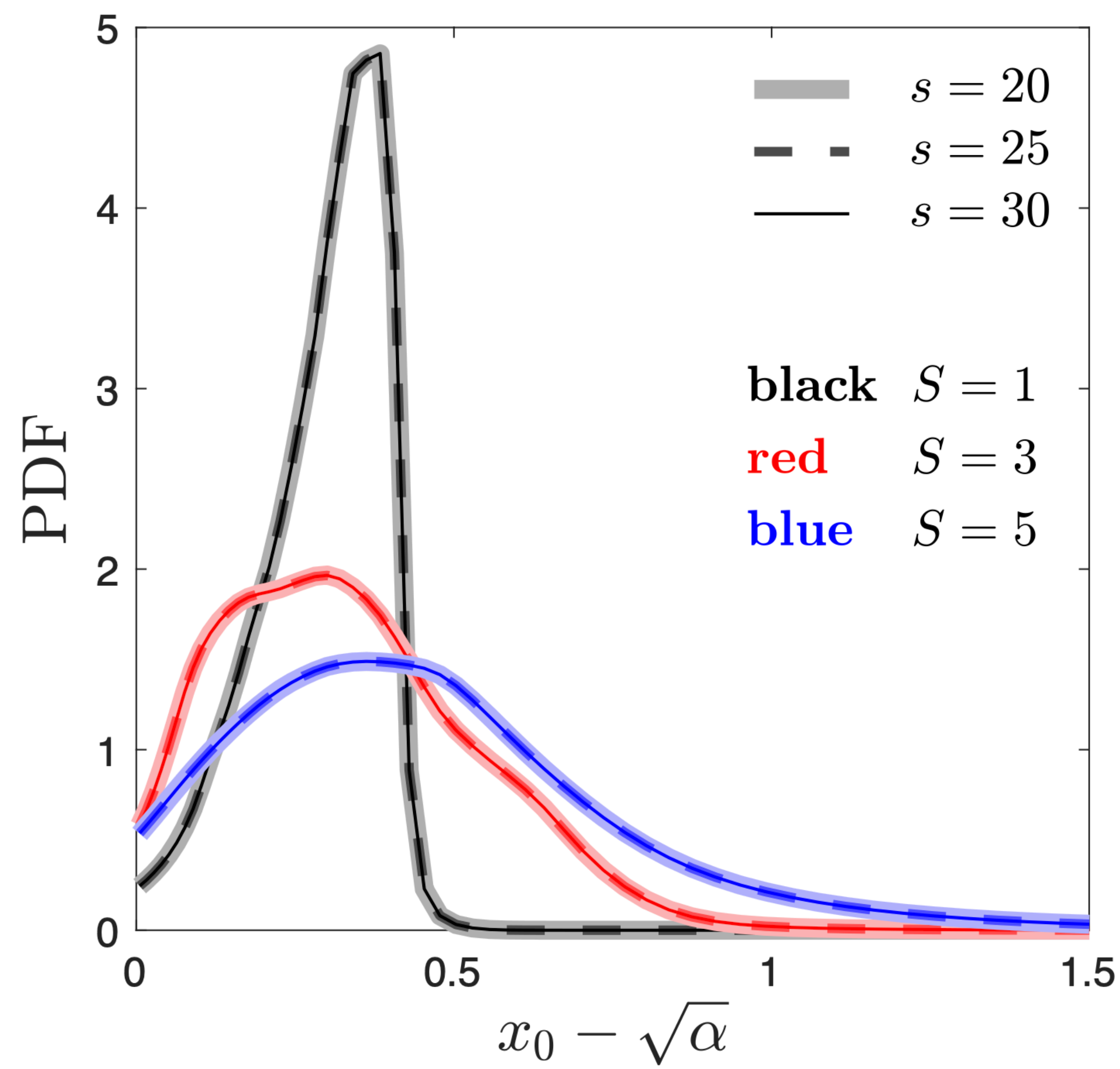
Inertial interval: large $S = s - m$



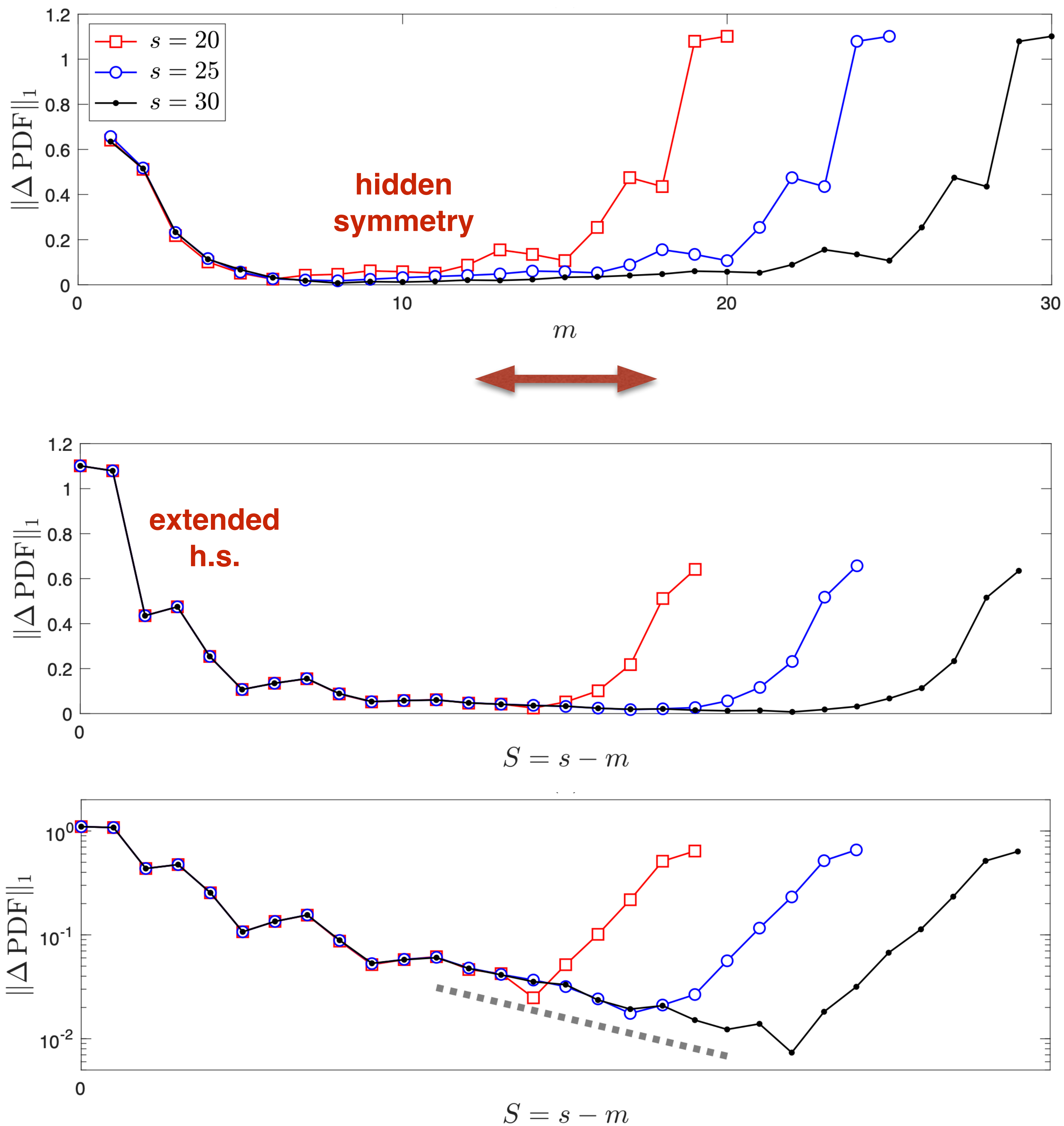
$s = 25,\ m = 6, \ldots, 10$

$s = 30,\ m = 6, \ldots, 15$

Dissipation range:



Deviation from the hidden-symmetric PDF for a multiplier $x_0 = \mathcal{X}_0[U^{(m)}]$



$$\|\Delta\, PDF\|_1 \propto 2^{-\zeta_D S},$$

$$\zeta_D \approx 0.25.$$

Extended self-similarity of structure functions

Recurrent expressions in rescaled variables: $$S_p(\ell_m) = u_0^p \int d\mu_p^{(m)}(x_0, x_{-1}, \ldots)$$

$$d\mu_p^{(m)} = \mathcal{L}_p^{(m-1)} \circ \mathcal{L}_p^{(m-2)} \circ \cdots \circ \mathcal{L}_p^{(1)}[d\mu_p^{(1)}]$$

Hidden symmetry in the inertial interval: $\mathcal{L}_p^{(m)} \approx \Lambda_p$ (independent of *m*)

Perron-Frobenius eigenmode: $$d\mu_p^{(m)} \approx C_p \lambda_p^m \, d\nu_p, \quad \Lambda_p[d\nu_p] = \lambda_p \, d\nu_p.$$

$$\boxed{S_p(\ell_m) \approx C_p u_0^p \left(\frac{\ell_m}{\ell_0}\right)^{\zeta_p}} \qquad \zeta_p = -\log_2 \lambda_p$$

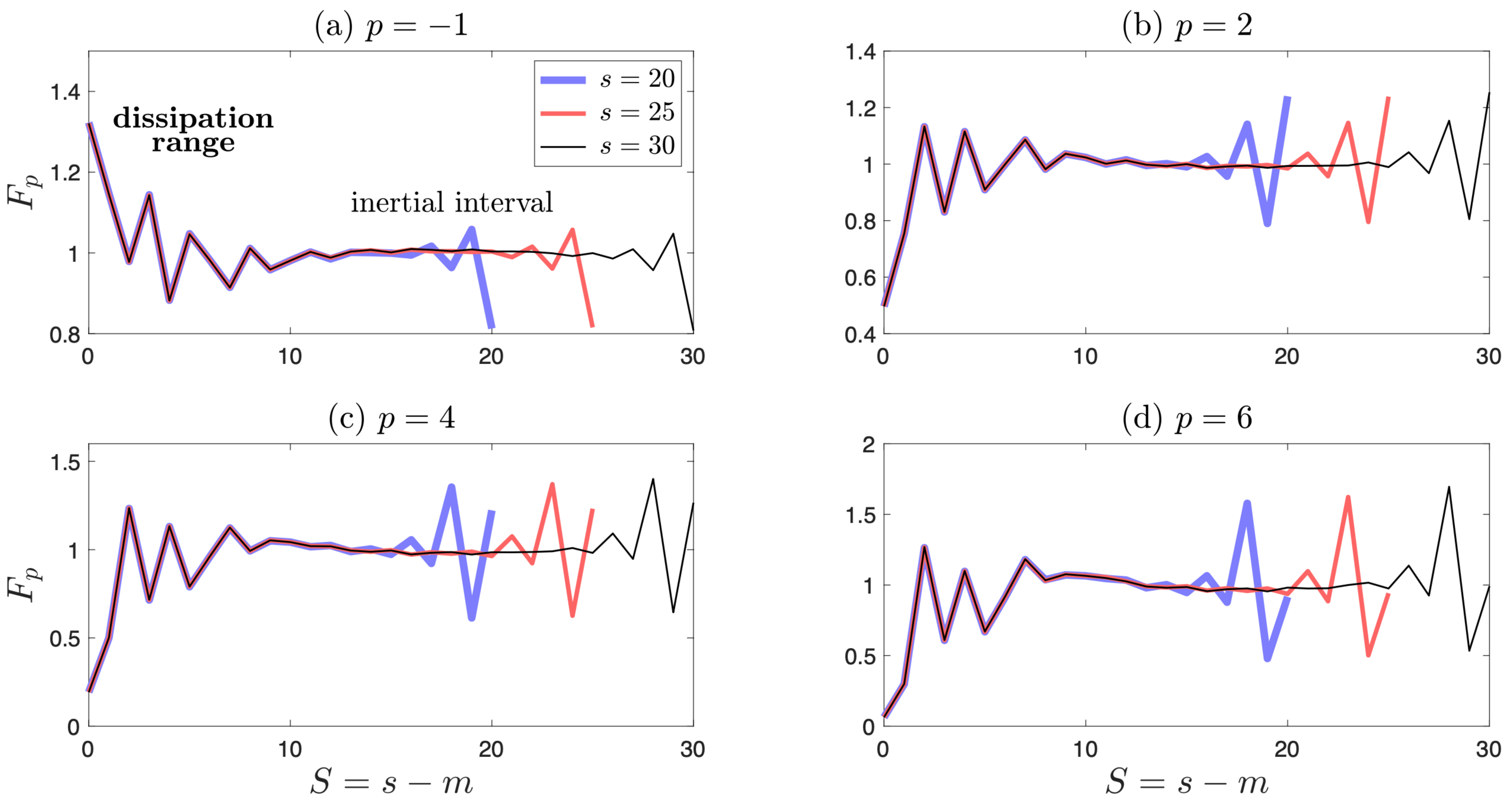
Extended hidden symmetry in the dissipation range: $\mathcal{L}_p^{(m)} \approx \Lambda_p^{(S)}$,

$$d\mu_p^{(m)} \approx C_p 2^{-\zeta_p m} 2^{-\zeta_p (S-d)} \Lambda_p^{(S+1)} \circ \Lambda_p^{(S+2)} \circ \cdots \circ \Lambda_p^{(d)}[d\nu_p]$$

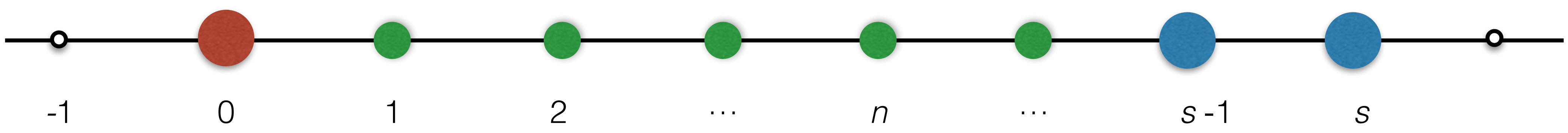
$$\boxed{S_p(\ell_m) \approx C_p u_0^p \left(\frac{\ell_m}{\ell_0}\right)^{\zeta_p} F_p\left(\frac{\ell_m}{\ell_s}\right)} \qquad F_p \to 1 \ \text{ for } \ \ell_m \gg \ell_s$$

Extended self-similarity of structure functions

$$\frac{1}{C_p u_0^p}\left(\frac{\ell_m}{\ell_0}\right)^{-\zeta_p} S_p(\ell_m) \approx F_p\left(\frac{\ell_m}{\ell_s}\right)$$



General form of viscous cutoffs with extended hidden symmetry



Equations of motion: $$\frac{du_n}{dt} = k_0 \mathcal{B}_n^{(s)}[u], \quad n = 1, \ldots, s$$

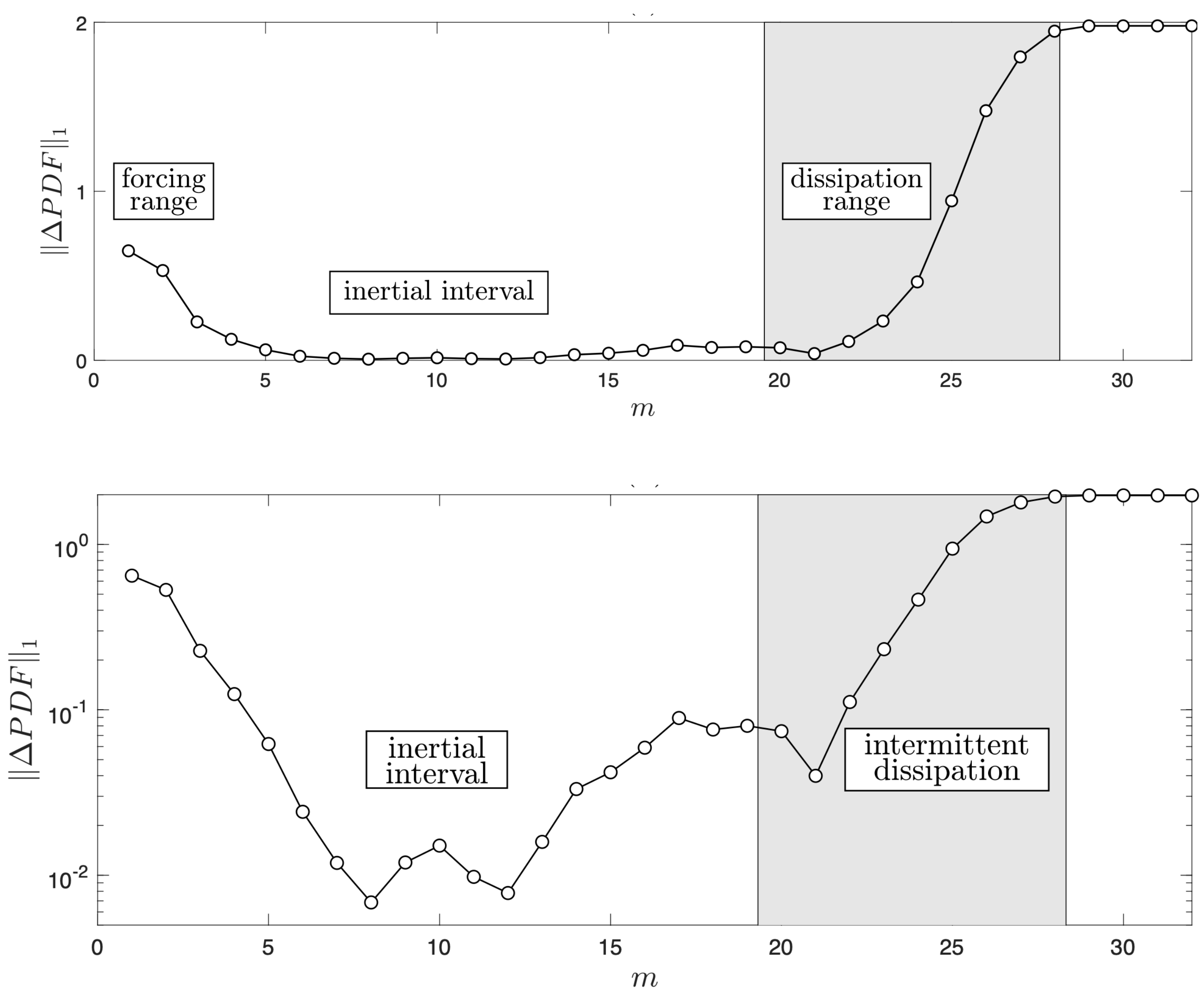
Quadratic terms:

$$\mathcal{B}_n^{(s)}[au] = a^2 \mathcal{B}_n^{(s)}[u], \quad a > 0$$ positive homogeneity

$$\mathcal{B}_n^{(s)}[u] = 2\mathcal{B}_{n-1}^{(s-1)}[u'], \quad u' = (u'_j)_{j\in\mathbb{Z}} = (u_{j+1})_{j\in\mathbb{Z}}$$ scaling relation

Conclusions I

- Inertial interval intermittency and multifractality is the consequence of hidden scale invariance.
- In the rescaled (self-similar) formulation, viscous terms are intermittent.
- Intermittency of dissipation yields a precise classification of scales:
  inertial interval (hidden symmetry is restored),
  intermittent dissipation range (gradually broken hidden symmetry)
  and far dissipation range (viscous forces are dominant).



Conclusions II

- Hidden symmetry extends to all small scales (inertial interval and dissipation range) for a specific class of viscous cutoff models.

- Statistical recovery of extended hidden symmetry yields anomalous self-similar form for structure functions at all small scales:

$$S_p(\ell_m) \approx C_p u_0^p \left(\frac{\ell_m}{\ell_0}\right)^{\zeta_p} F_p\left(\frac{\ell_m}{\ell_s}\right), \quad \zeta_p = -\log_2 \lambda_p$$

- Applications:
  - “Cleaner" analysis of the inertial interval (no intermittent dissipation range)
  - Computation of anomalous exponents from the data in the dissipation range
  - Understanding behavior of closures in turbulence models

    Eyink (lecture notes); Biferale, AAM, Parisi (2017);

    Biferale, Bonaccorso, Buzzicotti &Iyer (2019); Domingues Lemos (2022)
  - Extended hidden symmetry for the Navier-Stokes turbulence (e.g. LES)

Thank you!

alexei.impa.br

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